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GENERALIZED MODELS OF THE COSSERAT TYPE FOR FINITE DEFORMATION ANALYSIS OF THIN BODIES

L. I. Shkutin

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This work contains new invariant formulations of generalized models for analysis of the finite deformations of shell- and rod-shaped bodies with independent fields of finite displacements and rotations of material elements. They are obtained from a new invariant formulation of the nonlinear Cauchy model for a three-dimensional body with explicit isolation of local rotations. A two-dimensional generalized model of shell deformation is a consequence of the assumption of rigidly rotating transverse fibers. The three translational and two rotational degrees of freedom of a fiber form a system of five primary unknown variables of the generalized model. The absence of any independent rotation of the fiber relative to itself differentiates this system from the Cosserat axiomatic model of a deformable surface. A one-dimensional generalized model of a deformable rod is a consequence of the assumption of rigidly rotating cross-sections. The three translational and two rotational degrees of freedom of a cross-section form a system of six primary unknown variables of the generalized model. Its identity to the Cosserat axiomatic model of a deformable line with certain agreement between force parameters is established. In addition to the Cosserat axiomatic formulations, the generalized models include constitutive relations of a real material and equations for reconstruction of displacements, strains, and stresses in the volume of a real body.

The term *thin bodies* combines shells, plates, and rods. Such bodies are divided into two groups: *shell-shaped* and *rod-shaped* bodies. The first group includes shells, plates, and, if necessary, thin-walled rods, and the second one includes beams and rods with rigid cross-sections. A distinctive feature of a thin body is its small flexural rigidity in the direction of the small dimension. It can be strongly deflected under load, i.e., it undergoes strain with large gradients of displacements and rotations of material elements.

The geometrical features of thin bodies have made it possible to develop special mathematical models of their deformation that differ from the classical Cauchy models. Two approaches to modeling of deformations of thin bodies can be distinguished in the scientific literature: the *axiomatic* (direct) approach and the *approximation* approach. The first approach treats a shell as a *material surface* (two-dimensional object), considers a rod as a *material line* (one-dimensional object), determines the laws of their deformation under the action of *generalized* external and internal forces, and endows each material particle (a point of the object) not only with *position* (as in the Cauchy model) but also with *orientation* degrees of freedom. Axiomatic formulations of the deformation relations for flexible fibers and rods are known from the works of J. Bernoulli and L. Euler. They were generalized and extended to plates and shells by Cosserat et al. [1, 2]. The work of Ericksen and Truesdell [3] initiated a number of publications on constructing axiomatic models for deformation of media of any dimension with orientation degrees of freedom. A review of papers concerned with thin bodies can be found in [4, 5].

In the approximation approach, a shell and a rod are three-dimensional deformable Cauchy objects. A decrease in dimension is achieved by one or another approximation of the volume displacement field in the "thin" directions and by using the method of moments. Approximation formulations of the deformation equations for plates and rods are known from the works of A. Cauchy, S. Poisson, and G. Kirchhoff; they are extended to shells by H. Aron and A. Love. They have been improved and generalized up to the present time [6].

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As a result, both approaches give *moment* models for deformation of thin bodies. These models are defined in a space of a smaller dimension than that of physical space. Two-dimensional models correspond to shell-shaped bodies and one-dimensional models correspond to rod-shaped ones. While approximation models derived from the Cauchy model usually treat *displacements* as primary unknown variables, axiomatic models also assign *rotations* of local elements to primary unknown variables. This initial difference hinders comparative analysis of the competing models in nonlinear formulations.

This paper is devoted to approximation modeling of finite deformations of thin bodies and to comparison of the *generalized* models with the Cosserat axiomatic models. The concept of explicit isolation of the field of finite rotations in the nonlinear *Cauchy continuum* is used in the formulation of the initial three-dimensional problem of thin-body deformation. This concept was originated by G. Kirchhoff. In due time he went beyond the scope of linear elasticity theory by applying it to small prismatic volumes of a plate and a rod which experienced finite rotations as a result of deformation. The Kirchhoff analysis was extended to shells by A. Love, who retained the assumption that the deformation of a prismatic volume during its rotation is linear in character. Alumyaé et al. [7–9] gave a consistent nonlinear formulation of the *Kirchhoff–Love variant* for shells with isolation of finite rotations of linear elements. An extended model, which took into account transverse shears during finite rotations, was formulated by Simmonds et al. [10, 11]. The additional effect of transverse tension or compression is introduced into the model in [12, 13]. Shkutin [13] analyzed different variants of isolation of the finite-rotation field and determined that the variant of [10] is closest in its mathematical formulation to the *Cosserat model*. In the same work, a one-dimensional nonlinear model for spatial bending of a rod is developed by means of an approximation approach. This model is identical in its mathematical formulation to the *Cosserat model of a deformable line*.

The present work contains new invariant formulations of approximation models for nonlinear bending of shell- and rod-shaped bodies with explicit isolation of finite rotations of material elements. They are obtained owing to the introduction of generalized force and deformation tensors, which are *indifferent* to rigid rotations and *energetically conjugate* in the metrics of the rotating basis.

1. Isolation of Local Rotations in a Deformable Cauchy Body. Let an arbitrary Cauchy body in its initial (unstressed) state occupy a region (volume) G with a boundary (surface) A_ν in physical space. The subscript ν denotes surface orientation by the local unit normal vector $\mathbf{e}_\nu$. The region G is specified parametrically by a triplet of spatial coordinates t_I. They are also Lagrangian coordinates of a material particle (point) of the body. The position of such a point is given by the position vector $\mathbf{g}(t_I)$ in the initial configuration and by the vector $\mathbf{g}^+(t_I)$ in the deformed configuration. The triplet of vectors $\mathbf{g}_I^+ \equiv \partial_I \mathbf{g}^+$ forms a Lagrangian (material) coordinate basis $\mathbf{g}_I^+(\mathbf{g})$ with an initial value $\mathbf{g}_I(\mathbf{g})$. In addition, a convective basis $\mathbf{a}_I^0(\mathbf{g})$ with an initial value $\mathbf{a}_I(\mathbf{g})$, which rotates at a point as a rigid unity, is introduced at each point. Here and below, the upper-case and lower-case Latin letters in the subscripts have values of 1, 2, and 3 and values of 1 and 2, respectively; the tensor summation convention is used; the possible time dependence is not shown explicitly; ∂_I is the operator of partial differentiation with respect to the coordinate t_I.

The deformation of the Cauchy body in Lagrangian description is given by the mapping

$$\mathbf{g} \to \mathbf{g}^+, \qquad \mathbf{g}_I \to \mathbf{g}_I^+, \qquad \mathbf{g}^+ = \mathbf{g} + \mathbf{w}, \qquad \mathbf{g}_I^+ \equiv \partial_I \mathbf{g}^+, \tag{1.1}$$

where $\mathbf{w}(\mathbf{g})$ is the local displacement vector. In its deformed (instantaneous) state, the body is subjected to the action of external forces, which are distributed over its surface and volume. The volume external forces, together with inertial forces, are given by the density vector $\mathbf{p}^0(\mathbf{g})$ per initial unit volume, and the surface forces are given by the density vector $\mathbf{p}_\nu^0(\mathbf{g} \in A_\nu)$ per initial unit area. The internal stress field is introduced by the first Piola tensor $\mathbf{Z}_1(\mathbf{g})$.

The balance of external and internal forces for an instantaneous state of the Cauchy body can be expressed by the equation of virtual work (weak formulation):

$$\int_G (\mathbf{p}^0 \cdot \delta \mathbf{g}^+ - \partial W^0)\, dG + \int_{A_\nu} \mathbf{p}_\nu^0 \cdot \delta \mathbf{g}^+ \, dA_\nu = 0. \tag{1.2}$$

Here dG and dA_ν are the differentials of volume and surface; δ is the absolute variation operator; and ∂W^0

is the virtual strain energy density per initial unit volume, which is given by the equality

$$\partial W^0 = \mathbf{z}^I \cdot \delta \mathbf{g}_I^+ \quad (\mathbf{z}^I = \mathbf{g}^I \cdot \mathbf{Z}_1). \tag{1.3}$$

The triplet of contravariant stress vectors $\mathbf{z}^I(\mathbf{g})$ obeys a local equation for balance of moments:

$$\mathbf{z}^I \times \mathbf{g}_I^+ = 0. \tag{1.4}$$

The convective basis $\mathbf{a}_I^0(\mathbf{g})$ is introduced by the orthogonal mapping

$$\mathbf{a}_I^0 = \mathbf{a}_I \cdot \mathbf{\Theta}, \qquad \mathbf{\Theta} \equiv \mathbf{a}^I \mathbf{a}_I^0, \qquad \overline{\mathbf{\Theta}} \cdot \mathbf{\Theta} \equiv \mathbf{1}, \tag{1.5}$$

where $\mathbf{\Theta}(\mathbf{g})$ and $\overline{\mathbf{\Theta}}(\mathbf{g})$ are the mutually transposed tensors of finite rotation (rotators); $\mathbf{1}$ is the ordinary tensor. Differentiation of the vectors of the convective basis can be expressed by the transformation

$$\partial_I \mathbf{a}_J^0 = -\mathbf{a}_J^0 \cdot \mathbf{C}_I^0, \qquad -\overline{\mathbf{C}}_I^0 = \mathbf{C}_I^0 = \mathbf{1} \times \mathbf{c}_I^0 = \mathbf{c}_I^0 \times \mathbf{1}. \tag{1.6}$$

Here $\mathbf{C}_I^0(\mathbf{g})$ and $\mathbf{c}_I^0(\mathbf{g})$ are an antisymmetric tensor and the accompanying vector of the torsional flexure of the coordinate line t_I with initial values $\mathbf{C}_I(\mathbf{g})$ and $\mathbf{c}_I(\mathbf{g})$, respectively. The following rules of variation are valid:

$$\delta \mathbf{\Theta} = -\mathbf{\Theta} \cdot \mathbf{\Omega}, \qquad \delta \mathbf{a}_I^0 = -\mathbf{a}_I^0 \cdot \mathbf{\Omega}, \qquad -\overline{\mathbf{\Omega}} = \mathbf{\Omega} = \mathbf{1} \times \boldsymbol{\omega} = \boldsymbol{\omega} \times \mathbf{1}. \tag{1.7}$$

They introduce the spin $\mathbf{\Omega}(\mathbf{g})$ and vector $\boldsymbol{\omega}(\mathbf{g})$ of virtual rotation.

The second equality of (1.7) makes it possible to introduce the operator of corotational variation δ_0 such that

$$\delta_0 \mathbf{a}_I^0 \equiv \delta \mathbf{a}_I^0 + \mathbf{a}_I^0 \cdot \mathbf{\Omega} \equiv 0. \tag{1.8}$$

Its application to the vector $\mathbf{g}_I^+$ gives the mutually reversible equalities

$$\delta_0 \mathbf{g}_I^+ = \delta \mathbf{g}_I^+ + \mathbf{g}_I^+ \times \boldsymbol{\omega}, \qquad \delta \mathbf{g}_I^+ = \delta_0 \mathbf{g}_I^+ - \mathbf{g}_I^+ \times \boldsymbol{\omega}, \tag{1.9}$$

which are valid for any vector given in the convective basis. Replacement of the vector $\delta \mathbf{g}_I^+$ in (1.3) by its value in (1.9) and use of Eq. (1.4) lead to the following formula for the virtual strain energy:

$$\partial W^0 = \mathbf{z}^I \cdot \delta_0 \mathbf{g}_I^+. \tag{1.10}$$

It suggests a constructive representation of the material basis by the local transformation

$$\mathbf{g}_I^+ = \mathbf{a}_I^0 \cdot \mathbf{G}^+, \qquad \mathbf{G}^+ \equiv \mathbf{a}_0^I \mathbf{g}_I^+ \equiv \mathbf{a}_0^I \mathbf{a}_0^J G_{IJ}^+, \qquad G_{IJ}^+ \equiv \mathbf{g}_I^+ \cdot \mathbf{a}_J^0 \tag{1.11}$$

with the unknown distortion tensor $\mathbf{G}^+(\mathbf{g})$, which is not a metric tensor of the material basis.

The additive decomposition

$$\mathbf{G}^+ = \mathbf{G}^0 + \mathbf{W}, \quad \mathbf{G}^0 \equiv \overline{\mathbf{\Theta}} \cdot \mathbf{G} \cdot \mathbf{\Theta} = \mathbf{a}_0^I \mathbf{a}_0^J G_{IJ}, \quad G_{IJ} \equiv \mathbf{g}_I \cdot \mathbf{a}_J \tag{1.12}$$

introduces the deviation tensor

$$\mathbf{W} = \mathbf{G}^+ - \mathbf{G}^0 \equiv \mathbf{a}_0^I \mathbf{a}_0^J W_{IJ}, \qquad W_{IJ} \equiv G_{IJ}^+ - G_{IJ} \tag{1.13}$$

with convective components $W_{IJ}(\mathbf{g})$ and with zero value in the initial state. The equalities

$$\mathbf{W} = \mathbf{a}_0^I \mathbf{w}_I, \qquad \mathbf{w}_I \equiv \mathbf{a}_0^J W_{IJ} = \mathbf{g}_I^+ - \mathbf{g}_I \cdot \mathbf{\Theta}, \tag{1.14}$$

which are equivalent to (1.13), give a dyadic expansion of the deviation tensor and express it in terms of the displacement vector and the rotation tensor; $\mathbf{w}_I(\mathbf{g})$ are three deviation vectors.

The equalities

$$\delta_0 \mathbf{g}_I^+ = \mathbf{a}_I^0 \cdot \delta_0 \mathbf{G}^+ = \mathbf{a}_I^0 \cdot \delta_0 \mathbf{W} = \delta_0 \mathbf{w}_I, \tag{1.15}$$

follow from (1.11)–(1.14), where the quantities

$$\delta_0 \mathbf{G}^+ \equiv \mathbf{a}_0^I \mathbf{a}_0^J \delta G_{IJ}^+, \qquad \delta_0 \mathbf{W} \equiv \mathbf{a}_0^I \mathbf{a}_0^J \delta W_{IJ} \tag{1.16}$$

have the meaning of corotational variations of tensors. Their components are calculated from the formula

$$\delta W_{IJ} = \delta G^{+}_{IJ} = \mathbf{a}^0_J \cdot \delta_0 \mathbf{g}^+_I = \mathbf{a}^0_J \cdot (\delta \mathbf{g}^+_I + \mathbf{g}^+_I \times \boldsymbol{\omega}). \tag{1.17}$$

Substitution of (1.15) into (1.10) makes it possible to represent the quantity ∂W^0 as the equalities

$$\partial W^0 = \mathbf{z}^I \cdot \delta_0 \mathbf{w}_I = \overline{\mathbf{Z}} \cdot \cdot \delta_0 \mathbf{W} = Z^{IJ} \delta W_{IJ} \tag{1.18}$$

with the new stress tensor

$$\mathbf{Z} \equiv \mathbf{a}^0_I \mathbf{z}^I \equiv \mathbf{a}^0_I \mathbf{a}^0_J Z^{IJ}, \qquad Z^{IJ} \equiv \mathbf{z}^I \cdot \mathbf{a}^J_0, \tag{1.19}$$

which is related to the Piola tensor by the transformation

$$\mathbf{Z} = \overline{\boldsymbol{\Theta}} \cdot \mathbf{G} \cdot \mathbf{Z}_1. \tag{1.20}$$

In accordance with (1.18), the tensor $\mathbf{W}$ can serve as a measure of strain which is energetically conjugated with the stress tensor $\mathbf{Z}$. However, first it is necessary to eliminate the arbitrariness in the definition of the tensor $\mathbf{W}$. Orthogonal mapping (1.5) introduces into the Cauchy model an arbitrary rotator $\boldsymbol{\Theta}$, which has, as a vector, three degrees of freedom. Transformation (1.11), unlike the polar decomposition of the position gradient, defines the distortion tensor with rotational arbitrariness. Such arbitrariness can be eliminated by an additional condition of symmetry of the deviation tensor:

$$W_{JI} \equiv W_{IJ}. \tag{1.21a}$$

In this case, transformation (1.11) is equivalent to polar decomposition. The symmetric tensor contains six independent components, as does the triangular tensor. Therefore, instead of (1.21a), we can use the alternative variants

$$W_{32} \equiv W_{31} \equiv W_{21} \equiv 0; \tag{1.21b}$$

$$W_{23} \equiv W_{13} \equiv W_{12} \equiv 0. \tag{1.21c}$$

A tensor with an upper triangular matrix of components corresponds to the first variant, and a tensor with a lower triangular matrix corresponds to the second variant. In the case of variant (1.21a), transformations (1.5) and (1.11) isolate a certain averaged rotation of the instantaneous basis about the initial basis, while in variants (1.21b) and (1.21c,) they bring into coincidence the directions of individual basis vectors: $\mathbf{a}^0_3$ with $\mathbf{g}^+_3$, or $\mathbf{a}^0_1$ with $\mathbf{g}^+_1$. In isolating local rotations of thin bodies, variants (1.21b) and (1.21c) are preferable to variant (1.21a).

After the rotator is made to obey one of the variants of kinematic relations (1.21), the tensor $\mathbf{W}$, which is defined by equality (1.13), takes the meaning of a strain tensor which is energetically conjugated with the stress tensor (1.20). Both tensors are, by definition, indifferent to rigid rotations.

The resulting kinematic and dynamic equations must be supplemented by a formulation of relations for the deformative properties of the material. In particular, the purely mechanical processes of elastic and elastoplastic deformation of many construction materials are described by the linear *incremental* relation

$$\delta_0 \mathbf{Z} = \mathbf{D} \cdot \cdot \delta_0 \mathbf{W} \tag{1.22}$$

with the tensor of properties $\mathbf{D}$ (of the fourth rank), which can take into account the influence of the prehistory of loading. Incremental strain is calculated from increments of displacement and rotation vectors, by a formula of the form of (1.17). Relation (1.22) must obey condition (1.4), which ensures symmetry of the Cauchy stress tensor.

The formulation proposed is extended to a Cosserat deformable body by using the weak equation [5]

$$\int_G (\mathbf{p}^0 \cdot \delta \mathbf{w} + \mathbf{q}^0 \cdot \boldsymbol{\omega} - \partial W^0) dG + \int_{A_\nu} (\mathbf{p}^0_\nu \cdot \delta \mathbf{w} + \mathbf{q}^0_\nu \cdot \boldsymbol{\omega}) dA_\nu = 0 \tag{1.23}$$

with the virtual strain energy density

$$\partial W^0 = \overline{\mathbf{Z}} \cdot \cdot \delta_0 \mathbf{W} + \overline{\mathbf{Y}} \cdot \cdot \delta_0 \mathbf{V} = Z^{IJ} \delta W_{IJ} + Y^{IJ} \delta V_{IJ}. \tag{1.24}$$

In (1.23) and (1.24), $\mathbf{q}^0(\mathbf{g})$ is the vector of volume external and inertial moments; $\mathbf{q}_\nu^0(\mathbf{g} \in A_\nu)$ is the vector of surface external moments; $\mathbf{W}(\mathbf{g})$ is the metric strain tensor (1.13); $\mathbf{Z}(\mathbf{g})$ is the internal stress tensor (1.19); $\mathbf{Y}(\mathbf{g})$ is the internal moment tensor; $\mathbf{V}(\mathbf{g})$ is the bending tensor such that

$$\delta_0 \mathbf{V} \equiv \mathbf{a}_0^I \mathbf{a}_0^J \delta V_{IJ}, \qquad \delta V_{IJ} \equiv \mathbf{a}_I^0 \cdot \partial_I \omega. \tag{1.25}$$

Equations (1.23)–(1.25) contain, as primary unknown variables, the displacement vector $\mathbf{w}(\mathbf{g})$ and the rotation tensor $\Theta(\mathbf{g})$ which has, as a vector, three scalar degrees of freedom. Comparison of (1.23) and (1.24) with (1.2) and (1.18) shows that the above formulation of the Cauchy body model is degenerate relative to the Cosserat model, in which all external and internal moments are assumed to be absent and the rotation tensor is related to the displacement vector by conditions of the form of (1.21).

2. Model of the Cosserat Type for a Shell-Shaped Body. A spatial system of coordinates is connected with the basic surface A of a shell so that t_1 and t_2 are the internal parameters of the surface, and t_3 is the transverse coordinate. The volume occupied by the shell is usually bounded by an end surface A_3 and by two front surfaces A_n. The latter are given by the equation $t_3 = h_n$, where $h_1 \leqslant t_3 \leqslant h_2$ (h_1 and h_2 are known functions of a surface point or constant numbers). The surface A_N is oriented by the normal unit vector $\mathbf{e}_N(\mathbf{g} \in A_N)$. The differentials of the volume and the surfaces are determined by the equalities

$$dG \equiv J dt_3 dA, \quad dA \equiv a dt_1 dt_2, \quad J \equiv g/a, \quad dA_n \equiv j_n dA, \quad dA_3 \equiv j_3 dt_3 dC, \tag{2.1}$$

where $g(\mathbf{g})$ and $a(\mathbf{a})$ are the volume and surface Jacobians of the coordinate grid; $j_N(\mathbf{g} \in A_N)$ are the metric parameters of the surfaces; and dC is the differential of the boundary contour of the basic surface.

As a solid body, the shell is given by the equation $\mathbf{g} = \mathbf{a} + t_3 \mathbf{a}_3$, which expresses the volume position vector $\mathbf{g}(t_I)$ in terms of two surface vectors: a position vector $\mathbf{a}(t_i)$ and an orientation (normal) vector $\mathbf{a}_3(t_i)$. Two local bases, a volume basis $\mathbf{g}_I(\mathbf{g})$ and a surface basis $\mathbf{a}_I(\mathbf{a})$, are introduced in the initial state. They are related by the translation

$$\begin{gathered} \mathbf{g}^I = \mathbf{a}^I \cdot \mathbf{G}, \quad \mathbf{G} \equiv \mathbf{a}^I \mathbf{g}_I \equiv \mathbf{a}^I \mathbf{a}^J G_{IJ} = \mathbf{A} + t_3 \mathbf{B}, \quad \mathbf{A} \equiv \mathbf{a}^I \mathbf{a}_I \equiv \mathbf{a}^I \mathbf{a}^J A_{IJ}, \\ A_{IJ} \equiv \mathbf{a}_I \cdot \mathbf{a}_J, \quad \mathbf{a}_i \equiv \partial_i \mathbf{a}, \quad \mathbf{B} \equiv \mathbf{a}^i \mathbf{b}_i \equiv \mathbf{a}^i \mathbf{a}^j B_{ij}, \quad B_{ij} \equiv \mathbf{b}_i \cdot \mathbf{a}_j, \quad \mathbf{b}_i \equiv \partial_i \mathbf{a}_3. \end{gathered} \tag{2.2}$$

The surface tensors $\mathbf{A}(\mathbf{a})$ and $\mathbf{B}(\mathbf{a})$ determine the metrics and curvature of space in the vicinity of the basic surface.

Deformation of the shell-shaped body is studied within the framework of the Cauchy model and is described by local transformation (1.1). The volume external and inertial forces are given by the density vector $\mathbf{p}^0(\mathbf{g})$. The surface force field is divided into three fields, which are given at the end and front surfaces by the density vectors $\mathbf{p}_N^0(\mathbf{g} \in A_N)$. The virtual dynamic equation (1.2), which ensures the balance of external and internal forces in an instantaneous state, is formulated as

$$\int\limits_A \left[\int\limits_{h_1}^{h_2} (\mathbf{p}^0 \cdot \delta \mathbf{w} - \partial W^0) J dt_3 \right] dA + \int\limits_A \mathbf{p}_n^0 \cdot \delta \mathbf{w}^{(n)} j_n dA + \int\limits_C \left(\int\limits_{h_1}^{h_2} \mathbf{p}_3^0 \cdot \delta \mathbf{w} j_3 dt_3 \right) dC = 0, \tag{2.3}$$

where ∂W^0 is the volume density of virtual strain energy and $\delta \mathbf{w}^{(n)}$ is the value of the vector $\delta \mathbf{w}$ on the surface A_n.

The basic surface and its basis are deformed together with the shell: $\mathbf{a} \to \mathbf{a}^+(\mathbf{a})$ and $\mathbf{a}_I \to \mathbf{a}_I^+(\mathbf{a})$. A convective basis $\mathbf{a}_I^0(\mathbf{a})$ with the initial value $\mathbf{a}_I(\mathbf{a})$ is introduced on the deformed surface by the local transformation

$$\mathbf{a}_I^0 = \mathbf{a}_I \cdot \Theta, \qquad \Theta \equiv \mathbf{a}^I \mathbf{a}_I^0, \qquad \partial_3 \Theta \equiv 0 \tag{2.4}$$

with the tensor-rotator $\Theta(\mathbf{a})$.

With the assumption that the shell remains a thin body, its instantaneous state is given by the equation

$$\mathbf{g}^+ = \mathbf{a}^+ + t_3 \mathbf{a}_3^+, \quad \mathbf{a}_3^+ \equiv \mathbf{a}_3^0. \tag{2.5}$$

This equation corresponds to a linear approximation of the volume displacement field relative to the transverse coordinate:

$$\mathbf{w} = \mathbf{u} + t_3\mathbf{v}, \qquad \mathbf{u} \equiv \mathbf{a}^+ - \mathbf{a}, \qquad \mathbf{v} \equiv \mathbf{a}_3^0 - \mathbf{a}_3. \tag{2.6}$$

This relation gives the movement of a transverse material fiber of the shell as a rigid unit with translational displacement $\mathbf{u}(\mathbf{a})$ and rotational displacement $t_3\mathbf{v}(\mathbf{a})$.

The change in the volume basis due to displacement (2.6) is expressed by the transformation

$$\mathbf{g}_I^+ = \mathbf{a}_I^0 \cdot \mathbf{G}^+, \quad \mathbf{G}^+ \equiv \mathbf{a}_0^I \mathbf{g}_I^+ \equiv \mathbf{a}_0^I \mathbf{a}_0^J G_{IJ}^+ = \mathbf{A}^+ + t_3\mathbf{B}^+, \tag{2.7}$$

in which the surface tensors $\mathbf{A}^+(\mathbf{a})$ and $\mathbf{B}^+(\mathbf{a})$ are determined by the equalities

$$\begin{aligned} \mathbf{A}^+ &\equiv \mathbf{a}_0^I \mathbf{a}_I^+ \equiv \mathbf{a}_0^I \mathbf{a}_0^J A_{IJ}^+, \qquad A_{IJ}^+ \equiv \mathbf{a}_I^+ \cdot \mathbf{a}_J^0, \qquad \mathbf{a}_i^+ \equiv \partial_i \mathbf{a}^+, \\ \mathbf{B}^+ &\equiv \mathbf{a}_0^i \mathbf{b}_i \equiv \mathbf{a}_0^i \mathbf{a}_0^j B_{ij}^+, \qquad B_{ij}^+ \equiv \mathbf{b}_i^+ \cdot \mathbf{a}_j^0, \qquad \mathbf{b}_i^+ \equiv \partial_i \mathbf{a}_3^0. \end{aligned} \tag{2.8}$$

Formulas (1.13), (2.2), and (2.7) make it possible to represent the shell strain tensor as a linear function of the transverse coordinate:

$$\mathbf{W} \equiv \mathbf{G}^+ - \overline{\mathbf{\Theta}} \cdot \mathbf{G} \cdot \mathbf{\Theta} \equiv \mathbf{a}_0^I \mathbf{a}_0^J W_{IJ} = \mathbf{U} + t_3\mathbf{V}. \tag{2.9}$$

It introduces the surface tensors $\mathbf{U}(\mathbf{a})$ and $\mathbf{V}(\mathbf{a})$ of metric and torsional-flexural deformations:

$$\mathbf{U} \equiv \mathbf{A}^+ - \overline{\mathbf{\Theta}} \cdot \mathbf{A} \cdot \mathbf{\Theta} \equiv \mathbf{a}_0^i \mathbf{a}_0^J U_{iJ}, \quad U_{iJ} \equiv A_{iJ}^+ - A_{iJ}, \quad \mathbf{V} \equiv \mathbf{B}^+ - \overline{\mathbf{\Theta}} \cdot \mathbf{B} \cdot \mathbf{\Theta} \equiv \mathbf{a}_0^i \mathbf{a}_0^j V_{ij}, \quad V_{ij} \equiv B_{ij}^+ - B_{ij}. \tag{2.10}$$

They are both degenerate in the space of the convective basis.

Substitution of approximations (2.6) and (2.9) into (2.3) and (1.18) and integration with respect to the transverse coordinate lead to an equation of virtual work in a two-dimensional formulation:

$$\int_A [\mathbf{p} \cdot \delta\mathbf{u} + (\mathbf{a}_3^0 \times \mathbf{q}) \cdot \omega - \partial W] dA + \int_C [\mathbf{p}_3 \cdot \delta\mathbf{u} + (\mathbf{a}_3^0 \times \mathbf{q}_3) \cdot \omega] dC = 0 \tag{2.11}$$

with the surface density of virtual strain energy

$$\partial W \equiv \int_{h_1}^{h_2} \partial W^0 J dt_3 = \mathbf{x}^i \cdot \delta_0 \mathbf{a}_i^+ + \mathbf{y}^i \cdot \delta_0 \mathbf{b}_i^+ = \overline{\mathbf{X}} \cdot\cdot\, \delta_0 \mathbf{U} + \overline{\mathbf{Y}} \cdot\cdot\, \delta_0 \mathbf{V} = X^{iJ} \delta U_{iJ} + Y^{ij} \delta V_{ij}, \tag{2.12}$$

with the tensors of generalized internal forces and moments

$$\mathbf{X} \equiv \int_{h_1}^{h_2} \mathbf{Z} J dt_3 \equiv \mathbf{a}_I^0 \mathbf{a}_J^0 X^{IJ}, \ \mathbf{x}^i \equiv \mathbf{a}_0^i \cdot \mathbf{X}, \ \mathbf{Y} \equiv \int_{h_1}^{h_2} \mathbf{Z} J t_3 dt_3 \equiv \mathbf{a}_I^0 \mathbf{a}_J^0 Y^{IJ}, \ \mathbf{y}^i \equiv \mathbf{a}_0^i \cdot \mathbf{Y} \tag{2.13}$$

and with the vectors of generalized external forces and moments

$$\mathbf{p} \equiv \mathbf{p}_n^0 j_n + \int_{h_1}^{h_2} \mathbf{p}^0 J dt_3, \ \mathbf{p}_3 \equiv \int_{h_1}^{h_2} \mathbf{p}_3^0 j_3 dt_3, \ \mathbf{q} \equiv \mathbf{q}_n^0 j_n h_n + \int_{h_1}^{h_2} \mathbf{p}^0 J t_3 dt_3, \ \mathbf{q}_3 \equiv \int_{h_1}^{h_2} \mathbf{q}_3^0 j_3 t_3 dt_3. \tag{2.14}$$

The kinematic variables in (2.11) and (2.12) are varied according to the formulas

$$\begin{aligned} &\delta_0 \mathbf{U} = \mathbf{a}_0^i \mathbf{a}_0^J \delta U_{iJ}, \quad \delta U_{iJ} = \mathbf{a}_J^0 \cdot \delta_0 \mathbf{a}_i^+, \quad \delta_0 \mathbf{V} = \mathbf{a}_0^i \mathbf{a}_0^j \delta V_{ij}, \quad \delta V_{ij} = \mathbf{a}_j^0 \cdot \delta_0 \mathbf{b}_i^+, \quad \delta_0 \mathbf{a}_i^+ = \delta \mathbf{a}_i^+ + \mathbf{a}_i^+ \times \omega, \\ &\delta_0 \mathbf{b}_i^+ = \partial_i \omega \times \mathbf{a}_3^0, \qquad \delta_0 \mathbf{c}_i^0 = \partial_i \omega, \qquad \delta \mathbf{a}_i^+ = \partial_i \delta\mathbf{u}, \qquad \delta \mathbf{a}_3^+ = \delta \mathbf{a}_3^0 = \delta \mathbf{v} = \omega \times \mathbf{a}_3^0. \end{aligned} \tag{2.15}$$

Formula (1.6) is used to calculate derivatives of basis vectors.

Formulas (2.7) contain the equality $\mathbf{g}_3^+ = \mathbf{a}_3^0$, which orients the convective vector $\mathbf{a}_3^0$ in the direction of the material vector $\mathbf{g}_3^+$. Owing to (2.4) and (2.6), this equation is formulated as

$$\mathbf{v} = \mathbf{a}_3 \cdot \mathbf{\Theta} - \mathbf{a}_3. \tag{2.16}$$

This relation expresses definitively the independent vector of rotational displacement $\mathbf{v}(\mathbf{a})$ in terms of the tensor rotator $\mathbf{\Theta}(\mathbf{a})$. The latter, however, is not completely defined by relation (2.16). This is obvious, because the equality $\mathbf{a}_3^0 = \mathbf{g}_3^+$ does not orient the convective basis definitively: it leaves free the "drilling" rotation of the basis about the vector $\mathbf{g}_3^+$. This incomplete definition manifests itself clearly in the virtual formulation of relation (2.16):

$$\delta \mathbf{v} = -\mathbf{a}_3^0 \cdot \mathbf{\Omega} = \boldsymbol{\omega} \times \mathbf{a}_3^0. \tag{2.17}$$

This equality is fulfilled for an arbitrary value of the "drilling" component $\omega_3 \equiv \boldsymbol{\omega} \cdot \mathbf{a}_3^0 = \boldsymbol{\omega} \cdot \mathbf{g}_3^+$ of the rotation vector. Within the framework of the Cauchy model, this rotational degree of freedom must be eliminated, for example, by the trivial condition

$$\omega_3 \equiv \boldsymbol{\omega} \cdot \mathbf{a}_3^0 \equiv 0. \tag{2.18}$$

It is most naturally consistent with Eq. (2.11), in which external forces do not do any work in the virtual rotation ω_3.

In all those cases where an explicit dependence of the stress tensor (1.19) on the volume strain tensor (1.13) is known, formulas (2.13) take the meaning of generalized constitutive equations. With incremental dependences (1.22), these formulas give the generalized equations

$$\delta_0 \mathbf{X} = \mathbf{D}_1 \cdot\cdot \delta_0 \mathbf{U} + \mathbf{D}_2 \cdot\cdot \delta_0 \mathbf{V}, \quad \delta_0 \mathbf{Y} = \mathbf{D}_2 \cdot\cdot \delta_0 \mathbf{U} + \mathbf{D}_3 \cdot\cdot \delta_0 \mathbf{V}, \mathbf{D}_N \equiv \int_{h_1}^{h_2} \mathbf{D} J t_3^{N-1} dt_3.$$

The tensors $\delta_0 \mathbf{U}$ and $\delta_0 \mathbf{V}$ are calculated in terms of the independent incremental vectors $\delta \mathbf{u}$ and $\boldsymbol{\omega}$ from formulas of the form of (2.15).

For sufficiently smooth force fields, virtual equation (2.11) can be transformed to the Galerkin form

$$\int_A \{(\mathbf{p}+\nabla_i \mathbf{x}^i) \cdot \delta \mathbf{u} + [\mathbf{a}_3^0 \times \mathbf{q} + \mathbf{a}_i^+ \times \mathbf{x}^i + \nabla_i (\mathbf{a}_3^0 \times \mathbf{y}^i)] \cdot \boldsymbol{\omega}\} dA + \int_C [(\mathbf{p}_3 - e_{3i}\mathbf{x}^i) \cdot \delta \mathbf{u} + (\mathbf{q}_3 - e_{3i}\mathbf{y}^i) \cdot \delta \mathbf{v}] dC = 0, \tag{2.19}$$

where ∇_i is the operator of covariant differentiation on the initial basic surface and $e_{3i} \equiv \mathbf{e}_3 \cdot \mathbf{a}_i$ are components of the unit vector which is normal to the surface A_3 and the contour C. Global equality (2.19) generates local dynamic equations, which are valid at internal points of the basic surface, and boundary conditions at its contour. As is evident from equality (2.17), the vector $\delta \mathbf{v}$ has only two components in the convective basis. Therefore, the contour integral in (2.19) generates five scalar (kinematic or dynamic) conditions at the contour. The vectors $\delta \mathbf{v}$ and $\boldsymbol{\omega}$ are orthogonal and, under condition (2.18), interchangeable.

Two-dimensional equations (2.4), (2.8), and (2.10)–(2.18) form a weak formulation of the problem of deformation of the basic surface with independent fields of finite displacements and rotations of its material particle-points. Equalities (2.6) and (2.9) make it possible to reconstruct the volume fields of displacements and deformations of the shell. The stress field is calculated by volume constitutive relations (1.22) or by other relations. Extended system (2.4)–(2.18) formulates a generalized deformation model for a shell-shaped body with rigid transverse fibers and with separation of the finite-rotation field. Virtual equality (2.11) can be represented in the form of (2.19) and replaced by local dynamic equations. The generalized model in such a purely mechanical formulation has five primary kinematic parameters: three components of translational displacement and two components of rotation, which are related by a system of differential equations of the tenth order with respect to surface coordinates.

The Cosserat surface strain model is formulated by the weak equation

$$\int_A (\tilde{\mathbf{p}} \cdot \delta u + \tilde{\mathbf{q}} \cdot \omega - \partial W) dA + \int_C (\tilde{\mathbf{p}}_3 \cdot \delta \mathbf{u} + \tilde{\mathbf{q}}_3 \cdot \omega) dC = 0$$

with the surface density of the virtual strain energy

$$\partial W = \tilde{\mathbf{x}}^i \cdot \delta_0 \mathbf{a}_i^+ + \tilde{\mathbf{y}}^i \cdot \partial_i \omega.$$

Here $\boldsymbol{\omega}$ is the vector of virtual rotation with three degrees of freedom. The necessary and sufficient conditions for transformation of the Cosserat model into the model of a shell with rigid transverse fibers are given by the equalities

$$\tilde{\mathbf{p}} = \mathbf{p},\ \tilde{\mathbf{p}}_3 = \mathbf{p}_3,\ \tilde{\mathbf{x}}^i = \mathbf{x}^i,\ \tilde{\mathbf{q}} = \mathbf{a}_3^0 \times \mathbf{q},\ \tilde{\mathbf{q}}_3 = \mathbf{a}_3^0 \times \mathbf{q}_3,\ \tilde{\mathbf{y}}^i = \mathbf{a}_3^0 \times \mathbf{y}^i. \tag{2.20}$$

They show that the generalized model of a shell, unlike the Cosserat model, excludes transverse components of the external and internal moment vectors. These additional parameters inherent in the Cosserat model are polar reactions of the deformed surface to "drilling" microrotations. In a shell considered a Cauchy body, independent microrotations are not possible and the associated reactions are absent.

3. Model of the Cosserat Type for a Rod-Shaped Body. A spatial system of coordinates is connected with the basic line (contour) C_3 of a rod so that t_3 is the internal parameter of the line and t_1 and t_2 are the transverse coordinates. The volume occupied by the rod is usually bounded by a tubular surface A_3 and two end surfaces A_n. The latter are given by the equation $t_3 = l_n$, so that $l_1 \leqslant t_3 \leqslant l_2$ (l_1 and l_2 are constant numbers). The surface A_N is oriented by the normal unit vector $\mathbf{e}_N(\mathbf{g} \in A_N)$. The differentials of the volume and surfaces are defined by the equalities

$$dG \equiv J dt_3 dA, \quad dA_n \equiv dA \equiv dt_1 dt_2, \quad dA_3 \equiv j_3 dt_3 dC, \tag{3.1}$$

where $J(\mathbf{g})$ is the Jacobian of the coordinate grid; $j_3(\mathbf{g} \in A_3)$ is the metric parameter of the tubular surface; A is an arbitrary cross section; and C is its contour.

As a solid body the rod is given by the equation $\mathbf{g} = \mathbf{a} + t_i\mathbf{a}_i$, which expresses the volume position vector $\mathbf{g}(t_I)$ in terms of three contour vectors: a position vector $\mathbf{a}(t_3)$ and two orientation vectors $\mathbf{a}_i(t_3)$, which are normal to the basic line and to its tangential vector $\mathbf{a}_3 \equiv \partial_3\mathbf{a}$. In the initial state, two local bases are introduced: a volume basis $\mathbf{g}_I(\mathbf{g})$ and a contour (orthogonal) basis $\mathbf{a}_I(\mathbf{a})$, which are related to each other by the translation:

$$\begin{gathered}\mathbf{g}_I = \mathbf{a}_I \cdot \mathbf{G}, \quad \mathbf{G} \equiv \mathbf{a}^I\mathbf{g}_I \equiv \mathbf{a}^I\mathbf{a}^J G_{IJ} = \mathbf{A} + t_i\mathbf{B}_i, \quad \mathbf{A} \equiv \mathbf{a}^I\mathbf{a}_I \equiv \mathbf{a}^I\mathbf{a}^J A_{IJ}, \\ A_{IJ} \equiv \mathbf{a}_I \cdot \mathbf{a}_J, \quad \mathbf{a}_3 \equiv \partial_3\mathbf{a}, \quad \mathbf{B}_i \equiv \mathbf{a}^3\mathbf{b}_i \equiv \mathbf{a}^3\mathbf{a}^J B_{iJ}, \quad B_{iJ} \equiv \mathbf{b}_i \cdot \mathbf{a}_J, \quad \mathbf{b}_i \equiv \partial_3\mathbf{a}_i.\end{gathered} \tag{3.2}$$

The contour tensors $\mathbf{A}(\mathbf{a})$ and $\mathbf{B}_i(\mathbf{a})$ determine the metrics and curvature of space in the vicinity of the basic line.

Deformation of the rod-shaped body is studied within the framework of the Cauchy model and is described by local transformation (1.1). The volume external and inertial forces are given by the density vector $\mathbf{p}^0(\mathbf{g})$. The surface force field is divided into three fields given at the tubular and end surfaces by the density vectors $\mathbf{p}_N^0(\mathbf{g} \in A_N)$. The equation of virtual work (1.2) is formulated as

$$\int_{C_3} \Big[\int_A (\mathbf{p}^0 \cdot \delta\mathbf{w} - \partial W^0) J dA\Big] dt_3 + \int_{C_3} \Big(\int_C \mathbf{p}_3^0 \cdot \delta\mathbf{w} j_3 dC\Big) dt_3 + \int_{A_n} \mathbf{p}_n^0 \cdot \delta\mathbf{w}^{(n)} dA = 0. \tag{3.3}$$

Here ∂W^0 is the volume density of virtual strain energy and $\delta\mathbf{w}^{(n)}$ is the value of the vector $\delta\mathbf{w}$ at the end surface A_n.

The basic line and its basis are deformed together with the rod: $\mathbf{a} \rightarrow \mathbf{a}^+(\mathbf{a})$ and $\mathbf{a}_I \rightarrow \mathbf{a}_I^+(\mathbf{a})$. The local transformation

$$\mathbf{a}_I^0 = \mathbf{a}_I \cdot \Theta, \quad \Theta \equiv \mathbf{a}^I\mathbf{a}_I^0, \quad \partial_i\Theta \equiv 0 \tag{3.4}$$

with the tensor rotator $\Theta(\mathbf{a})$ on the deformed line introduces a convective basis $\mathbf{a}_I^0(\mathbf{a})$ with an initial value $\mathbf{a}_I(\mathbf{a})$. Under the assumption that the rod remains a thin body, its instantaneous state is given by the equation

$$\mathbf{g}^+ = \mathbf{a}^+ + t_i\mathbf{a}_i^+, \quad \mathbf{a}_i^+ \equiv \mathbf{a}_i^0, \tag{3.5}$$

to which corresponds a linear approximation of the volume field of displacements relative to the transverse coordinates

$$\mathbf{w} = \mathbf{u} + t_i\mathbf{v}_i, \quad \mathbf{u} \equiv \mathbf{a}^+ - \mathbf{a}, \quad \mathbf{v}_i \equiv \mathbf{a}_i^0 - \mathbf{a}_i. \tag{3.6}$$

This relation gives the movement of the material cross section of the rod as a rigid unit with translational displacement $\mathbf{u}(\mathbf{a})$ and rotational displacement $t_i\mathbf{v}_i(\mathbf{a})$.

The change in the volume basis due to displacement (3.6) is expressed by the transformation

$$\mathbf{g}_I^+ = \mathbf{a}_I^0 \cdot \mathbf{G}^+, \quad \mathbf{G}^+ \equiv \mathbf{a}_0^I \mathbf{g}_I^+ \equiv \mathbf{a}_0^I \mathbf{a}_0^J G_{IJ}^+ = \mathbf{A}^+ + t_i \mathbf{B}_i^+, \tag{3.7}$$

in which the contour tensors $\mathbf{A}^+(\mathbf{a})$ and $\mathbf{B}_i^+(\mathbf{a})$ are defined by the equalities

$$\begin{aligned} \mathbf{A}^+ &\equiv \mathbf{a}_0^I \mathbf{a}_I^+ \equiv \mathbf{a}_0^I \mathbf{a}_0^J A_{IJ}^+, \quad A_{IJ}^+ \equiv \mathbf{a}_I^+ \cdot \mathbf{a}_J^0, \quad \mathbf{a}_3^+ \equiv \partial_3 \mathbf{a}^+, \\ \mathbf{B}_i^+ &\equiv \mathbf{a}_0^3 \mathbf{b}_i^+ \equiv \mathbf{a}_0^3 \mathbf{a}_0^J B_{iJ}^+, \quad B_{iJ}^+ \equiv \mathbf{b}_i^+ \cdot \mathbf{a}_J^0, \quad \mathbf{b}_i^+ \equiv \partial_3 \mathbf{a}_i^0. \end{aligned} \tag{3.8}$$

Formulas (1.13), (3.2), and (3.7) make it possible to represent the strain tensor of the rod as a linear function of the transverse coordinates:

$$\mathbf{W} \equiv \mathbf{G}^+ - \overline{\mathbf{\Theta}} \cdot \mathbf{G} \cdot \mathbf{\Theta} \equiv \mathbf{a}_0^3 \mathbf{a}_0^J W_{3J} = \mathbf{U} + t_i \mathbf{V}_i. \tag{3.9}$$

This function introduces the contour tensors $\mathbf{U}(\mathbf{a})$ and $\mathbf{V}_i(\mathbf{a})$ of metric and torsional-flexural deformations:

$$\begin{aligned} \mathbf{U} &\equiv \mathbf{A}^+ - \overline{\mathbf{\Theta}} \cdot \mathbf{A} \cdot \mathbf{\Theta} \equiv \mathbf{a}_0^3 \mathbf{a}_0^J U_{3J}, \qquad U_{3J} \equiv A_{3J}^+ - A_{3J}, \\ \mathbf{V}_i &\equiv \mathbf{B}_i^+ - \overline{\mathbf{\Theta}} \cdot \mathbf{B}_i \cdot \mathbf{\Theta} \equiv \mathbf{a}_0^3 \mathbf{a}_0^J V_{iJ}, \qquad V_{iJ} \equiv B_{iJ}^+ - B_{iJ}. \end{aligned} \tag{3.10}$$

They are both degenerate in the space of the convective basis.

Substitution of approximations (3.6) and (3.9) into (3.3) and (1.18) and integration with respect to the cross section lead to the equation of virtual work in a one-dimensional formulation:

$$\int_{C_3} (\mathbf{p} \cdot \delta \mathbf{u} + \mathbf{q} \cdot \omega - \partial W) dt_3 + \mathbf{p}_n \cdot \delta \mathbf{u}^{(n)} + \mathbf{q}_n \cdot \omega^{(n)} = 0 \tag{3.11}$$

with the contour density of virtual strain energy,

$$\partial W \equiv \int_A \partial W^0 J dA = \mathbf{x}^3 \cdot \delta_0 \mathbf{a}_3^+ + \mathbf{y}_i^3 \cdot \delta_0 \mathbf{b}_i^+ = \overline{\mathbf{X}} \cdot \cdot \delta_0 \mathbf{U} + \overline{\mathbf{Y}}_i \cdot \cdot \delta_0 \mathbf{V}_i = X^{3J} \delta U_{3J} + Y_i^{3J} \delta V_{iJ}, \tag{3.12}$$

with the tensors of generalized internal forces and moments,

$$\mathbf{X} \equiv \int_A \mathbf{Z} J dA \equiv \mathbf{a}_I^0 \mathbf{a}_J^0 X^{IJ}, \ \mathbf{x}^3 \equiv \mathbf{a}_0^3 \cdot \mathbf{X}, \ \mathbf{Y}_i \equiv \int_A \mathbf{Z} J t_i dA \equiv \mathbf{a}_I^0 \mathbf{a}_J^0 Y^{IJ}, \ \mathbf{y}_i^3 \equiv \mathbf{a}_0^3 \cdot \mathbf{Y}_i, \tag{3.13}$$

and with the vectors of generalized external forces and moments,

$$\mathbf{p} \equiv \int_A \mathbf{p}^0 J dA + \int_C \mathbf{p}_3^0 j_3 dC, \quad \mathbf{q}_n \equiv \mathbf{a}_i^0 \times \int_{A_n} \mathbf{p}_n^0 t_i dA, \quad \mathbf{p}_n \equiv \int_{A_n} \mathbf{p}_n^0 dA, \quad \mathbf{q} \equiv \mathbf{a}_i^0 \times \Big(\int_A \mathbf{p}^0 J t_i dA + \int_C \mathbf{p}_3^0 j_3 t_i dC \Big). \tag{3.14}$$

The kinematic variables in (3.11) and (3.12) vary according to the formulas

$$\begin{gathered} \delta_0 \mathbf{U} = \mathbf{a}_0^3 \mathbf{a}_0^J \delta U_{3J}, \quad \delta U_{3J} = \mathbf{a}_J^0 \cdot \delta_0 \mathbf{a}_3^+, \quad \delta_0 \mathbf{V}_i = \mathbf{a}_0^3 \mathbf{a}_0^J \delta V_{iJ}, \quad \delta V_{iJ} = \mathbf{a}_J^0 \cdot \delta_0 \mathbf{b}_i^+, \\ \delta_0 \mathbf{a}_3^+ = \delta \mathbf{a}_3^+ + \mathbf{a}_3^+ \times \omega, \quad \delta_0 \mathbf{b}_i^+ = \partial_3 \omega \times \mathbf{a}_i^0, \quad \delta_0 \mathbf{c}_i^0 = \partial_3 \omega, \quad \delta \mathbf{a}_3^+ = \partial_3 \delta \mathbf{u}, \quad \delta \mathbf{a}_i^+ = \delta \mathbf{a}_i^0 = \delta \mathbf{v}_i = \omega \times \mathbf{a}_i^0. \end{gathered} \tag{3.15}$$

It is seen that the variations of metric and torsional–flexural deformations are definitively expressed in terms of two independent virtual vectors: $\delta\mathbf{u}$ and ω. Their components give six degrees of freedom for deformation of the basic line of the rod.

When the explicit dependence of the stress tensor (1.19) on the volume strain tensor (1.13) is known, formulas (3.13) take the meaning of generalized constitutive equations. In particular, incremental relations (1.22) become, by means of (3.13), the generalized equations

$$\delta_0 \mathbf{X} = \mathbf{D}_0 \cdot \cdot \delta_0 \mathbf{U} + \mathbf{D}_j \cdot \cdot \delta_0 \mathbf{V}_j, \quad \delta_0 \mathbf{Y}_i = \mathbf{D}_i \cdot \cdot \delta_0 \mathbf{U} + \mathbf{D}_{ij} \cdot \cdot \delta_0 \mathbf{V}_j,$$

$$\mathbf{D}_0 \equiv \int_A \mathbf{D} J dA, \quad \mathbf{D}_i \equiv \int_A \mathbf{D} J t_i dA, \quad \mathbf{D}_{ij} \equiv \int_A \mathbf{D} J t_i t_j dA.$$

The tensors $\delta_0\mathbf{U}$ and $\delta_0\mathbf{V}_i$ are calculated in terms of the independent incremental vectors $\delta\mathbf{u}$ and ω using formulas of the type of (3.15).

For sufficiently smooth force fields, virtual equation (3.11) is written in the Galerkin form

$$\int_{C_3}[(\mathbf{p}+\partial_3\mathbf{x}^3)\cdot\delta\mathbf{u}+(\mathbf{q}+\mathbf{a}_3^+\times\mathbf{x}^3+\partial_3\mathbf{y}^3)\cdot\omega]dt_3$$

$$+[(\mathbf{p}_n-e_{n3}\mathbf{x}^3)\cdot\delta\mathbf{u}+(\mathbf{q}_n-e_{n3}\mathbf{y}^3)\cdot\omega]_{t_3=l_n}=0,\quad \mathbf{y}^3\equiv\mathbf{a}_i^0\times\mathbf{y}_i^3, \tag{3.16}$$

where $e_{n3}\equiv\mathbf{e}_n\cdot\mathbf{a}_3$ are components of the unit vector that is normal to the end surface A_n. Global equality (3.16) generates local dynamic equations, which are valid at interior points of the basic line, and conditions at boundary points.

One-dimensional equations (3.4), (3.8), and (3.10)–(3.15) give a weak formulation of the problem of deformation of the basic line with independent fields of finite displacements and rotations of its material particle-points of the line. Equalities (3.6) and (3.9) reconstruct the volume field displacement and strain of the rod. The stress field is calculated by the volume constitutive relations (1.22) or by other relations. Extended system (3.4)–(3.15) formulates a generalized model of deformation of a rod-shaped body with rigid cross-sections and with separation of the finite-rotation field. The virtual equality (3.11) can be written in the form of (3.16) and replaced by local dynamic equations. In this purely mechanical formulation, the generalized model contains six primary kinematic parameters: three translational displacement components and three rotation components, which are related by a system of differential equations of the twelfth order with respect to the contour coordinate.

The *Cosserat line* strain model is formulated by the weak equation [5]

$$\int_{C_3}(\tilde{\mathbf{p}}\cdot\delta\mathbf{u}+\tilde{\mathbf{q}}\cdot\omega-\partial W)dt_3+\tilde{\mathbf{p}}_n\cdot\delta\mathbf{u}^{(n)}+\tilde{\mathbf{q}}_n\cdot\omega^{(n)}=0$$

with the contour density of virtual strain energy

$$\partial W=\tilde{\mathbf{x}}^3\cdot\delta_0\mathbf{a}_3^++\tilde{\mathbf{y}}^3\cdot\partial_3\omega.$$

The conditions of transformation of the Cosserat model into the model of a rod with rigid cross-sections are given by the equalities

$$\tilde{\mathbf{p}}\equiv\mathbf{p},\quad \tilde{\mathbf{p}}_n\equiv\mathbf{p}_n,\quad \tilde{\mathbf{x}}^3\equiv\mathbf{x}^3,\quad \tilde{\mathbf{q}}\equiv\mathbf{q},\quad \tilde{\mathbf{q}}_n\equiv\mathbf{q}_n,\quad \tilde{\mathbf{y}}^3\equiv\mathbf{a}_i^0\times\mathbf{y}_i^3. \tag{3.17}$$

They show the meaning of the axiomatic force parameters of the Cosserat model as applied to a rod-shaped body.

The above invariant formulations of the equations of deformation of shell- and rod-shaped Cauchy bodies are generalized models of the *Cosserat type*. In addition to the axiomatic Cosserat models, they give a construction algorithm for generalized (two- and one-dimensional) constitutive relations which are consistent with the local properties of the body material and contain equations for reconstruction of the volume displacement, strain, and stress fields in a thin body.

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